DATE OF EXAM: 25.02.2019 **Solution**
SUBJECT NAME: Analysis II - MIDTERM Exam - Semester II

1. *Let U be an open subset of X, and let $x \in U$. Is $U \setminus \{x\}$ open in X? Justify your answer.*

   Solution: We claim that $U \setminus \{x\}$ is open in X. Let $y \in U \setminus \{x\}$ be a arbitrary element. It is easy to observe that $y \in U$. Since U is open subset X, then there exits $r > 0$ such that $B(y,r) \subset U$. We will have two cases, either $x \notin B(y,r)$ or $x \in B(y,r)$. If $x \notin B(y,r)$ then $B(y,r) \subset U \setminus \{x\}$. If $x \in B(y,r)$ then choose $r_0 = \frac{d(y,x)}{2}$, therefore $B(y,r_0) \subset U \setminus \{x\}$. Hence $U \setminus \{x\}$ is an open set.

   $\square$

2. *Let $f, g : X \to \mathbb{R}_u$ be continuous functions, and let $C = \{x \in X : f(x) = g(x)\}$. Prove that C is closed.*

   Solution: Consider a sequence of points $\{x_n\} \in C$ converging to $x \in X$. To prove C is closed, it is enough to prove that $x \in C$. Since $f(x_n) = g(x_n)$ for all n and f, g are continuous, then we have

   $$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$

   Therefore $x \in C$. Thus C contains its limit points. Hence C is closed.

   $\square$

3. *Prove that a discrete metric space is complete.*

   Solution: Let $\{x_n\}$ be a Cauchy sequence, then for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > n_0$. Now take $\epsilon = \frac{1}{2}$, then there exist $n_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{1}{2}$ for all $n, m > n_1$. Since d is discrete metric, we have x_n is a constant for all $n > n_1$. Therefore $\{x_n\}$ converges in discrete metric space.

   $\square$

4. *Define* $C = \{x \in X : B_r(x) \textit{ is uncountable for every } r > 0\}$. Prove that C is a closed set.

   Solution: Consider a sequence of points $\{x_n\} \in C$ converging to $x \in X$. To prove C is closed, it is enough to prove that $x \in C$. Since $x_n \to x$, for arbitrary $\epsilon > 0$, $B_\epsilon(x)$ contains atleast one element of the sequence $\{x_n\}$. Say $x_m \in B_\epsilon(x)$, using our assumption, we can find an $r_0 > 0$ such that $B_{r_0}(x_m) \subset B_\epsilon(x)$. Since $B_{r_0}(x_m)$ is uncountable, therefore $B_\epsilon(x)$ is uncountable. Hence $x \in C$.

   $\square$

5. *Let $\tilde{d}(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$ for all $A, B \subset X$. State whether the following statements are ture or false. Justify your answer.*
   (i) If $A \cap B \neq \emptyset$, then $\tilde{d}(A,B) = 0$. (ii) If $\tilde{d}(A,B) = 0$, then $A \cap B \neq \emptyset$. (iii) $(P(X), \tilde{d})$ is a metric space.

   Solution: (i). If $A \cap B \neq \emptyset$, then we can find $a \in A \cap B$ such that $d(a,a) = 0$. This implies that $\tilde{d}(A,B) = 0$. Therefore (i) is true.
   (ii). Let $A = (0,1)$ and $B = (1,2)$, $\tilde{d}(A,B) = 0$, but $A \cap B = \emptyset$. Therefore (ii) is false.
   (iii).Let $A, B \in P(X)$, assume that $A \neq B$, but $A \cap B \neq \emptyset$. From (i), we will get $\tilde{d}(A,B) = 0$. Therefore $(P(X), \tilde{d})$ is not a metric space. (iii) is false.

   $\square$

6. *Let D be a dense subset of X such that every Cauchy sequence in D converges in X. Prove that X is complete.*

   Solution: Let $\{x_n\}$ be a Cauchy sequence in X. Since D is dense in X. For each $n \in \mathbb{N}$ there is some y_n such that $d(x_n, y_n) < \frac{1}{n}$. Thus

$$d(y_m, y_n) \leq d(y_m, x_m) + d(x_m.x_n) + d(x_n, y_n) < \frac{1}{m} + d(x_m, x_n) + \frac{1}{n}.$$

   For every pair of positive integers m and n. Therefore $\{y_n\}$ is also a Cauchy sequence in D. By assumption $\{y_n\}$ converges to $x \in X$.

$$d(x_n, x) \leq d(x_n, y_n) + d(y_n, x) < \frac{1}{n} + d(y_n, x).$$

   Thus $x_n \to x$. Hence X is complete.

   $\square$

7. *Prove that* $\mathbb{R}^n_u \setminus \mathbb{Q}^n, n > 1$, is connected.

   Solution: We can find the proof in the book 'Topology of Metric Spaces' by S. Kumaresan. Example 5.1.30, Page - 112 to 113.

   $\square$

8. *Let X be a countable metric space, and let $\#X \geq 1$. Prove that X is connected if and only if $\#X = 1$*

   Solution: Assume $\#X = 1$, every metric space with one element is connected. Therefore X is connected. Conversely, assume that X is connected. Suppose X has atleast two elements x_0, x_1 with $x_0 \neq x_1$. Define $f : X \to [0, 1]$ by

$$f(x) = \frac{d(x, x_0)}{d(x, x_0) + d(x, x_1)} \quad \text{for all} \quad x \in X$$

   Thus f is continuous and we have $f(x_0) = 0$ and $f(x_1) = 1$. Since X is connected and the continuous image of connected set is connected set is connected. Therefore $f(X) = [0, 1]$, Since $[0, 1]$ is uncountable, implies X is uncountable. This is a contradiction. Therefore $\#X = 1$. $\square$

9. *Let $f : X \to Y$ be a continuous function, and let*

$$Graph(f) = \{(x, f(x)) : x \in X\}.$$

   Prove that Graph(f), with respect to the metric inherited from the product metric, is homeomorphic to X.

   Solution: Define $F : X \to \text{Graph}(f)$ by $F(x) = (x, f(x))$. It is easy to verify that F is bijective. Now, $g : X \to X$ defined by $g(x) = x$ is continuous and by assumption f is continuous, implies that F is continuous. $F^{-1} : \text{Graph}(f) \to X$ maps $F^{-1}((x, f(x))) = x$. F^{-1} can be obtained as the restriction of the from $X \times Y$ to X, to the subspace G of $X \times Y$. Since projections are continuous, therefore F^{-1} is continuous.

   $\square$